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Closure systems over effect algebras

Sistemas de cierre sobre algebras de efecto

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ABSTRACT

The present paper is an attempt to introduce the closure systems over effect algebras. At first, we will define closure systems over effect algebras, and for arbitrary set $ U $ and arbitrary subset S of all functions from U to an effect algebra L we will obtain the closure system containing S. Then, we will define the base of this closure system, and for arbitrary subset S of all functions from U to an effect algebra L we will obtain the base of this closure system.

Keywords: Closure Systems, Closure operator, Effect Algebra, Base.

RESUMEN

El presente artículo es un intento de introducir los sistemas de cierre sobre las álgebras de efectos. Primero definiremos sistemas de cierre sobre álgebras de efectos y para el conjunto arbitrario $ U $ y el subconjunto arbitrario S de todas las funciones de U a un álgebra de efectos L obtendremos el sistema de cierre que contiene S. Luego definiremos la base de este sistema de cierre y para un subconjunto arbitrario S de todas las funciones desde U hasta un álgebra de efectos L obtendremos la base de este sistema de cierre.

Palabras clave: Sistemas de cierre, Operador de cierre, Álgebra de efectos, Base.

1. INTRODUCTION

Closure systems have an important role in almost all parts of mathematics and computer science, specially databases, data analysis and management of data. The first people who introduced the concept of closure system were (Abramsky & Jung, 1994), and (Belohlavek, 2001). L closure systems (with L being a partially ordered set) are introduced in (Abramsky & Jung, 1994) and (Belohlavek, 2001) and (Nola et al., 2002) and (Lu & Wang, 2011) This concept generalizes the ordinary subsets of U or equivalently, characteristic functions $A : U \to \{0, 1\}$ to $A : U \to L$ that are referred to as L -sets. The partially ordered set L can have different algebraic structures such as Boolean algebra, MV-algebra, BL-algebra or effect algebra Effect algebra was first introduced by (Foulis & Bennett, 1994) An effect algebra is a partial algebraic structure, originally formulated as an algebraic base for unsharp quantum measurements.

In this article we try to introduce the concept closure system over effect algebra.

This paper is organized as following: In section 2, we will present the content that we need in the paper. In section 3, we will introduce closure system over effect algebra. In section 4, we will introduce bases of a closure system.

2. MATERIALS AND METHODS

For research in this article, library studies and a collection of articles and books have been used.

Preliminaries

Effect algebras are abstract generalizations of the unit interval $[0,1] \subseteq \mathbb{R}$.

$[0,1] \subseteq \mathbb{R}$ carries a partial addition: the sum of two elements may or may not lie in the $[0,1] \subseteq \mathbb{R}$ again. Furthermore, it has a minimal and a maximal element, and complements with respect to the maximal element. We capture the algebraic structure of [0,1] in the notion of an effect algebra

2.1. Definition

(Foulis & Bennett, 1994) An effect algebra L is a structure $(L, +, ', 0, 1)$ consisting of a set L with two special elements $0, 1$, unary operation $'$ and a partially defined binary operation $+$ on $L \times L$ satisfying

the following conditions for every $x, y, z \in L$

1. Commutativity: if $x + y$ is defined, then so is $y + x$, and $x + y = y + x$
2. Associativity: if $x + y$ and $(x + y) + z$ are defined, then so are $y + z$ and $x + (y + z)$, and $(x + y) + z = x + (y + z)$.
3. Zero: 0+a is always defined and equals a
4. Orthocomplement: for each $x \in L, x'$ is the unique element for which $x + x' = 1$
5. Zero-one law: if $x + 1$ is defined, then $x = 0$.

When $x + y$ exists, we say that x is orthogonal to y and will denote this by $x \perp y$.

On an effect algebra L one can define a partial order $\leq$ as

(O_1) $\quad x \leq y \Leftrightarrow \exists\, t \in L\ (x + t = y)$

Below we list a useful set of properties of effect algebras (Foulis & Bennett, 1994) and (Dvurecenskij & Pulmannova, 2000).

2.2. Proposition

Let L be an effect algebra and a, b and c be elements of L:

$$(\mathrm{P}_1)\quad \mathrm{a}'' = \mathrm{a}. \tag{1}$$
$$(\mathrm{P}_2)\quad 1' = 0,\ 0' = 1. \tag{2}$$
$$(\mathrm{P}_3)\quad \mathrm{a} \perp 0\,, \mathrm{a} + 0 = 1. \tag{3}$$
$$(\mathrm{P}_4)\quad \mathrm{a} \perp 1 \Leftrightarrow \mathrm{a} = 0. \tag{4}$$
$$(\mathrm{P}_5)\ \mathrm{a} + \mathrm{b} = 0 \Leftrightarrow \mathrm{a} = \mathrm{b} = 0. \tag{5}$$
$$(\mathrm{P}_6)\ \mathrm{a} \perp \mathrm{b} \Leftrightarrow \mathrm{a} \leq \mathrm{b}'. \tag{6}$$
$$(\mathrm{P}_7)\ \mathrm{a} \leq \mathrm{b} \Leftrightarrow \mathrm{b}' \leq \mathrm{a}'. \tag{7}$$
$$(\mathrm{P}_8)\ \mathrm{a} + \mathrm{c} = \mathrm{b} + \mathrm{c} \Rightarrow \mathrm{a} = \mathrm{b}. \tag{8}$$
$$(\mathrm{P}_9)\ \mathrm{a} + \mathrm{c} \leq \mathrm{b} + \mathrm{c} \Rightarrow \mathrm{a} \leq \mathrm{b}. \tag{9}$$
$$(\mathrm{P}_{10})\ \mathrm{a} \leq \mathrm{b} \Rightarrow \mathrm{a} \perp (\mathrm{a} + \mathrm{b}')'. \tag{10}$$
$$(\mathrm{P}_{11})\ \mathrm{a} \leq \mathrm{b} \Rightarrow \mathrm{a} \perp (\mathrm{a} + \mathrm{b}')' = \mathrm{b}. \tag{11}$$
$$(\mathrm{P}_{12})\ \mathrm{a} + \mathrm{b} = \mathrm{c} \Leftrightarrow \mathrm{a}' = \mathrm{b} + \mathrm{c}'. \tag{12}$$

2.3. Lemma

In (P_8) *and* (P_9) of the previous proposition, if $c \perp a\,, c \perp b$, it can be concluded that

$$(P_8{}')\quad a = b \Rightarrow a + c = b + c \tag{13}$$
$$(P_9{}')\quad a \leq b \Rightarrow a + c \leq b + c \tag{14}$$

Proof:

$(P_8{}')$ It is clear.

$(P_9{}')\ Since\ a \leq\ b\ \ by\ (\ O_1)\ there\ exists\ \ t \in\ L,$

$b = a + t, so\ \ c + b =\ c + a + t\ .\ Therefore\ ,\ a + c \leq b + c.$

A closure system S on a set L is a set of subsets of L containing L and any intersection of subsets of S. In the sequel we can see the definition of closure system over effect algebras.

If U is a set and L is an effect algebra, then we define L^U as the set of all functions from U to L

An L - closure operator in U is a function $F: L^U \longrightarrow L^U$ such that for all $A, B \in L^U$ the following properties hold

- $A \subseteq F(A)$
- If $A \subseteq B$ then, $F(A) \subseteq F(B)$
- $F\big(F(A)\big) = A$

3. RESULTS AND DISCUSSION

If L be an effect algebra and U be a nonempty set. For $S \subseteq L^U$, the closure system which containing S is S_∞ and $Rdu_+(\ldots Rdu_+(Rdu_\wedge(Rdu_+(S))))$ or $Rdu_\wedge(\ldots Rdu_\wedge(Rdu_+(Rdu_\wedge(S))))$ is a base of S_∞ .

Closure Systems Over Effect Algebras

3.1. Definition

If L is an effect algebra, U is a nonempty set and $S \subseteq L^U$ we call S a closure system over L if the following conditions are satisfied:

- S is closed under $\wedge -$ intersections, i.e. if $A_i \in S$,then $\wedge A_i \in S$.
- S is closed under summations, i.e. for all $A \in S$ and $a \in L$, if $a \perp A$, then $a + A \in S$.

Here, $a + A$, $\wedge A_i$ are defined by

$$(a + A)(u) = a + A(u), \ \ (\wedge A_i) = \wedge A_i(u). \tag{15}$$

In which $a \perp A$ means , $a \perp A(u)$ for all $u \in U$.

Obviously, the set L^U is a closure system (the largest one) and we can easily see that an intersection of an arbitrary system of closure systems is a closure system. Hence, it follows from classical results (Davey, 2002) that for every set $S \subseteq L^U$ there exists the least closure system $\overline{S}$ containing S, namely the intersection of all closure systems that contain S.

3.2. Definition

A base of a closure system $T \subseteq L^U$ is a set $S \subseteq L^U$ such that

- $\bar{S} = T$.
- $\bar{P} \neq T$, for every $P \subseteq S$.

3.3. Definition

For $S \subseteq L^U$, we put:

$$\hat{S} = \{\wedge A : \emptyset \neq A \subseteq S\} \tag{16}$$
$$S_+ = \{a + A : a \in L , A \in S, a \perp A\} \tag{17}$$

in the unit interval $[0,1] \subseteq \mathbb{R}$ We mean the phrase $\frac{a}{b}$ for the rational number, i.e. $\frac{a}{b} \in \mathbb{Q}$ \$ and $+$ is partial addition in $[0,1] \subseteq \mathbb{R}$, i.e. $\frac{a}{b} + \frac{c}{d}$ may or may not lie in the $[0,1] \subseteq \mathbb{R}$ again. In the next example (a, b) or $[a, b]$ means a set of all functions whose values are in the (a, b) or $[a, b]$, in which (a, b) and $[a, b]$ are open and closed intervals in $\mathbb{R}$

3.4. Example

consider the unit interval $[0,1] \subseteq \mathbb{R}$, $U = \{1\}$ and

(a) $S_1 = \left\{\frac{1}{5}, \frac{1}{8}, \frac{2}{7}\right\}$

Hence $\widehat{S_1} = S_1$, $S_{1+} = \left[\frac{1}{5}, 1\right] \cup \left[\frac{1}{8}, 1\right] \cup \left[\frac{2}{7}, 1\right] = [\frac{1}{8}, 1]$

(b) $S_1 = \left\{\frac{1}{10}, \left(\frac{1}{7}, \frac{3}{4}\right), \frac{8}{9}\right\}$

Then $\widehat{S_2} = S_2$, $S_{2+} = \widehat{S}_{2+}$=$[\frac{1}{10}, 1]$

It is well known that the closure systems and closure operators are cryptomorphic [1] mathematical structures||.

the closure operator associated with a closure system defines the closure of a subset E of L as the least closed set containing E and the closure system

associated with a closure operator is the family of its fixed points.

in the next theorem we show that functions

$\widehat{(.)}$, $(.)_+ : \mathcal{P}(L^U) \longrightarrow \mathcal{P}(L^U)$ such that for all

$S \subseteq L^U, \widehat{(.)}\ (S) = \hat{S},\ (.)_+(S) = S_+$ are closure operators.

3.5. Theorem

Let L be an effect algebra and U be a nonempty set, then $\widehat{(.)}$, $(.)_+$

are closure operators on L^U.

Proof. It is clear that for all $S \subseteq L^U$, $S \subseteq \hat{S}$.

Let $S_1 \subseteq S_2$. Since for all $T \subseteq S_1, T \subseteq S_2$, we have $\widehat{S_1} \subseteq \widehat{S_2}$. Now consider $\hat{\hat{S}} = \{\wedge T : T \subseteq \hat{S}\}$

[I]Informally a structure Γ on a set E can be seen as a set of axioms bearing on mathematical objects(operations, maps, families of subsets,..) defined on E Let Γ, Γ' be two structures defined on E. They are cryptomorphic if there exist maps between the objects of the two structures which transform any assertion true in one of these structures into an assertion true in the other one. For instance, the structure of Boolean algebra is cryptomorphic with the structure of Boolean ring (Nicoletti et al., 1988), for a more precise formulation.

Since for all $S \subseteq L^U$, $S = 0 + S$. Hence $S \subseteq S_+$. Now consider $a + A \in S_{+_+}$ hence $a \in S_+$, $a \perp t$, for all t ∈A . Since $A \in S_+$, therefore $A = a' + A'$, which $A' \in S$, $a' \perp t$ for all $t \in A'$. So $a + A = a + a' + A' \in S_+$.

Now let $S_1 \subseteq S_2$, $A \in S_1$ and $a + A \subseteq S_{1+} \subseteq S_{2+}$, so $(.)_+$ is a closure operator.

We may define the direct product of a family of effect algebras as follows Assume that U is a nonempty set and $L_u,\ u \in U$ are effect algebras. The direct product of family $\{ L_u\colon\ u \in U \}$ of effect algebras , denoted by $\prod_{u\in U} L_u$ is an effect algebra in which partially binary operation $+$ and unary operation $(-)'$ defined pointwise. In the other words $\prod_{u\in U} L_u$ is the set of all functions $f\colon U \longrightarrow \bigcup_{u\in U} L_u$ such that $f(u) \in L_u$ for all $u \in U$ with the partially defined binary operation $+$and the unary operation $(-)'$defined by*:*

if for all $u \in U, f(u) \perp g(u), (f + g)(u) = f(u) + g(u),\ f'(u) = f(u)'$. The least and the greatest element of $\prod_{u\in U} L_u$ are the functions $0{,}1\colon U \longrightarrow \bigcup_{u\in U} L_u$ such that

$0(u) = 0L_u, 1(u) = 1L_u.$

Order in direct product is pointwise, i.e. $\prod_{u\in U} a_u \leq \prod_{u\in U} b_u$ iff $a_u \leq b_u$.

Since operations in direct product of effect algebras are pointwise, therefore direct product of effect algebras is an effect algebra.

3.6. Proposition

The direct product of effect algebras is an effect algebra

Like MV-algebras (Nola et al., 2002) for every effect algebra L and nonempty set U, L^u is direct product of the family

$\{ L_u\colon\ u \in U \}$ where $L_u = L$, for all $u \in U$.

Based on what has been said, the following examples are subsets of direct product of $[0,1]$

The following example shows that in general, even if L is a chain $\widehat{S_+} \nsubseteq \hat{S}_+ , \hat{S}_+ \nsubseteq \widehat{S_+}$

In the next example by $(\frac{1}{1000}, \frac{1}{20})$ we mean a function like f from $\{0,1\}$ to $[0,1]$ in which $f(0) = \frac{1}{1000}, f(1) = \frac{1}{20}$

3.7. Example

Consider the unit interval $[0, 1] \subseteq \mathbb{R}$,

$U = \{0, 1\}$

(1) $S_1 = \{\left(\frac{1}{1000}, \frac{1}{20}\right), \left(\frac{1}{40}, \frac{1}{300}\right)\}$

Obviously $\left(\frac{1}{1000} + \frac{1}{1000}, \frac{1}{1000} + \frac{1}{20}\right) \wedge \left(\frac{1}{40}, \frac{1}{300}\right) = (\frac{1}{1000} + \frac{1}{1000}, \frac{1}{300}) \in \widehat{S_{1+}}$, but $(\frac{1}{1000} + \frac{1}{1000}, \frac{1}{300}) \notin \widehat{S}_{1+}$. So in general

$\widehat{S_{1+}} \nsubseteq \widehat{S}_{1+}$

(2) $S_2 = \left\{\left(\frac{1}{1000}, \frac{99}{100}\right), \left(\frac{9}{10}, \frac{1}{100}\right)\right\}$

Therefore $\frac{99}{100} + \left(\frac{1}{1000}, \frac{1}{100}\right) = \left(\frac{99}{100} + \frac{1}{1000}, 1\right) \in \widehat{S}_{2+}$.

We claim $\left(\frac{99}{100}+\frac{1}{1000},1\right) \notin \widehat{S_{2+}}$, if $\left(\frac{99}{100}+\frac{1}{1000},1\right) = (t+\frac{1}{1000}, t+\frac{99}{100}) \wedge (t^{'}+\frac{9}{10}, t^{'}+\frac{1}{100}) = (t+\frac{1}{1000}, t^{'}+\frac{1}{100})$

Where $0 \le t^{'} \le \frac{1}{10}$, which is a contradiction.

3.8. Lemma

Let L be an effect algebra and U be a nonempty set, $S \subseteq L^U$, $\overline{S_0} = S$ and $\overline{S_{\iota+1}} = \widehat{\overline{S_\iota}} \cup \overline{S_{\iota}}_+$ then $\bigcup \overline{S_\iota}$ is a closure system.

Proof. Let $\in \bigcup S_i$. So for some i, $A \in S_i$. Now we consider $a \in L$ in which, $a \perp A$, then $a + A \in \overline{S_{\iota}}_+ \subseteq \overline{S_{\iota+1}} \subseteq \bigcup \overline{S_\iota}$.

Now let $\{A_i : i \in I\} \subseteq \bigcup \overline{S_\iota}$, since $\overline{S_\iota} \subseteq \overline{S_{\iota+1}}$, there exists

j such that $\{A_i : i \in I\} \subseteq \overline{S_J}$. Therefore $\wedge A_i \in \widehat{\overline{S_J}} \subseteq \overline{S_{J+1}} \subseteq \bigcup \overline{S_\iota}$.

3.9. Corollary

Let L be an effect algebra and U be a nonempty set, $S \subseteq L^U$**,** then $\bar{S} = \bigcup \overline{S_\iota}$ where, , $\overline{S_0} = S$ and $\overline{S_{\iota+1}} = \widehat{\overline{S_\iota}} \cup \overline{S_{\iota}}_+$.

Proof. Since for all , $\widehat{\overline{S_\iota}} \subseteq \bar{S}$, $\overline{S_{\iota}}_+ \subseteq \bar{S}$. Therefore

$\bigcup \overline{S_\iota} \subseteq \bar{S}$. On the other hand , according to Lemma, because $\bigcup \overline{S_\iota}$ is a closure system and $\bar{S}$ is the smallest closure system, therefore $\bar{S} \subseteq \bigcup \overline{S_\iota}$.

3.10. Definition

Let L be an effect algebra and U be a nonempty set, for each $S \subseteq L^U$ **,** we define

$$S_{\oplus} = \{a_u + A(u) : a_u \in L , A \in S , a_u \perp A(u)\} \tag{18}$$

The next example shows that in general $S_+ \neq S_{\oplus}$.

3.11. Example

Consider $= [0,1], U = \{0,1\}$ **,**

$S = \{(\frac{1}{1000}, \frac{99}{100}), (\frac{9}{10}, \frac{1}{100})\}$, so $S_+ = \{(\frac{1}{100}+t, \frac{99}{100}+t) : 0 \le t \le \frac{1}{100}\} \cup \{(\frac{9}{10}+t, \frac{1}{100}+t) : 0 \le t \le \frac{1}{10}\}$.

But $S_{\oplus} = \{(\frac{1}{1000}+t, \frac{99}{100}+t') : 0 \le t \le \frac{999}{1000}, 0 \le t' \le \frac{1}{100}\} \cup \{(\frac{9}{10}+t, \frac{1}{100}+t') : 0 \le t \le \frac{1}{10}, 0 \le t' \le \frac{99}{100}\}$

3.12. Definition

Let L be an effect algebra and U be a nonempty set and $S \subseteq L^U$, we define

$$S_\infty = \widehat{\widehat{\hat{S}_{+_{+}}}}_{\dots}^{\dots} \tag{19}$$

3.13. Lemma

Let L be an effect algebra and U be a nonempty set and $S \subseteq L^U$, then $\bar{S} = S_\infty$.

Proof. Since for all

$A \subseteq S_\infty$. $\widehat{A}, A_+ \subseteq S_\infty$, so S_∞ is a closure system . Therefore $\bar{S} \subseteq S_\infty$. Since $\hat{S} \subseteq \overline{S_1}, \hat{S}_+ \subseteq \overline{S_2}, \dots$ we have $S_\infty \subseteq \bar{S}$.

3.14. Corollary

Let L be an effect algebra and U be a nonempty set and $S \subseteq L^U$, then

$$S_\infty = \widehat{\widehat{\widehat{S_{+}}_{+}}_{+}}_{\dots}^{\dots} \tag{20}$$

Proof. Since $\widehat{\widehat{\widehat{S_{+}}_{+}}_{+}}_{\dots}^{\dots}$ is a closure system , $\overline{S} \subseteq \widehat{\widehat{\widehat{S_{+}}_{+}}_{+}}_{\dots}^{\dots}$, and since $S_+ \subseteq \overline{S_1}, \widehat{S_+} \subseteq \overline{S_2}, \dots$, we have $\widehat{\widehat{\widehat{S_{+}}_{+}}_{+}}_{\dots}^{\dots} \subseteq \overline{S}$

$\widehat{(.)}$- *Base* , $(.)_+$- *Base* *and* $\overline{(.)}$- *Base*

In this section, we are giving some algorithms to construct different bases. First some basic concepts are reviewed

3.15. Definition

(Belohlavek & Konecny, 2016) Let L be an effect algebra and U be a nonempty set. For $S \subseteq L^U$, $\widehat{(.)}$- base of $\widehat{S}$ is a set

$S_0 \subseteq L^U$ such that

- $\widehat{S_0} = \widehat{S}$
- $\widehat{T} \neq \widehat{S}$ for every $T \subset S_0$

Let L be an effect algebra and U be a nonempty set. For $S \subseteq L^U$ we consider the set of elements in S minimal with respect to $\widehat{(.)}$, i.e.
$Rdu_\wedge(S) = \{A \in S : A \notin \widehat{S - \{A\}}\}$

The following theorem is a folklore in lattice theory. Note also that the theorem follows from the results on bases in domain theory on irreducibility (Abramsky & Jung, 1994) and (Gierz et al., 2004) and (Mundici, 2007).

3.16. Theorem

Let L be an effect algebra and U be a nonempty set. For every finite $S \subseteq L^U$, $Rdu_{\wedge}(S)$ is a unique $\widehat{(.)}$- base of $\hat{S}$.

3.17. Definition

Let L be an effect algebra and U be a nonempty set. Let R denote the binary relation on L^U defined by

$B_1 \ R \ B_2$ if and only if, for some $a \in L$, $a \perp B_1$ and $B_2 = a + B_1$.

3.18. Lemma

Let R be defined as above. Then

1. R is reflexive, antisymmetric and transitive
2. $B_1 \ R \ B_2$ implies, ${B_2}_+ \subseteq {B_1}_+$.

   Proof.
   1. Since $0 + A = A$, R is reflexive. Now let for some $a , a' \in L$ and $B_1 , B_2 \in S$, $a + B_1 = B_2$, $a' + B_2 = B_1$. So for all $u \in U$, $a + B_1(u) = B_2(u), a' + B_2(u) = B_1(u)$. Hence $a + a' = 0$, so by (P_5), $a = a' = 0$, therefore R is antisymmetric.
      Let for some $a , a' \in L$ and $B_1 , B_2 , B_3 \in S$.
      $B_2 = a + B_1$, $B_3 = a' + B_2$, so $B_3 = a' + a + B_2$. Thus R is transitive.
   2. Since $B_1 \ R \ B_2$ for some $a \in L$,
      $B_2 = a + B_1$. Now consider $a' + B_2 \in {B_2}_+$, clearly
      $a' + B_2 = a' + a + B_1 \in {B_1}_+$.

3.19. Definition

Let L be an effect algebra and U be a nonempty set. For every finite $S \subseteq L^U$, $Rdu_+(S)$ denote the set of all minimal elements in S with respect to R , i.e.

$$Rdu_+(S) = \{B \in S : \ B_1 \ R \ B \ \textit{implies} \ B_1 = B \ \textit{for some} \ B_1 \in S\} \qquad (21)$$

3.20. Definition

Let L be an effect algebra and U be a nonempty set. For $S \subseteq L^U$, $(.)_+$- base of S S_+ is a set

$S_0 \subseteq L^U$ such that

- ${S_0}_+ = S_+$

- $T_+ \neq S_+$ for every $T \subset S_0$

3.21. Theorem

Let L be an effect algebra and U be a nonempty set. For any finite $S \subseteq L^U$, $Rdu_+(S)$ is unique

$(.)_+$- base of S_+

Proof.

We prove at first $(Rdu_+(S))_+ = S_+$. By definition, we have

$(Rdu_+(S))_+ = \{a + B : a \perp B, B \in Rdu_+(S)\}$. Since

$Rdu_+(S) \subseteq S$, $(Rdu_+(S))_+ \subseteq S_+$. Now let $B \in S_+$.

So, $B = a_1 + B_1$ for some $B_1 \in S$, $a_1 \in L$.

Since S is finite, there exists $B_2 \in Rdu_+(S)$ such that

$B_2 \; R \; B_1$,i.e. $B_1 = a_2 + B_2$, for some $a_2 \in L$. Hence

$B = a_1 + B_1 = a_1 + (a_2 + B_2) = (a_1 + a_2) + B_2 \in (Rdu_+(S))_+$

If $Rdu_+(S)$ is redundant, then there exists $B \in Rdu_+(S)$ such that , $B \in (Rdu_+(S) - \{B\})_+$. Therefore $B = a + B_1$ such that $B_1 \neq B$, which is a contradiction.

Now let T is another $(.)_+$- base . Then since $Rdu_+(S) \subseteq S_+ = T_+$. For each $B \in Rdu_+(S)$, there exists $a_1 \in L$, $B_1 \in T$ such that $B = a_1 + B_1$,i.e. $B_1 \; R \; B$. As $B_1 \in T \subseteq T_+ = S_+$, there exists $a_2 \in L$ and $B_2 \in Rdu_+(S)$ such that $B_1 = a_2 + B_2$,i.e. $B_2 \; R \; B_1$ .Due to transitivity of R . Since $B, B_2 \in Rdu_+(S)$ and since B is minimal in S, we obtain $B = B_2$. Observe that by previous Lemma we have $B_2 \subseteq B_1 \subseteq B$ therefore $B = B_1 \in T$. Therefore $Rdu_+(S) \subseteq T$. since $Rdu_+(S)$ is a $(.)_+$- base, we must have because $Rdu_+(S) = T$, otherwise T is redundant.

3. CONCLUSION

Lemma Let L be an effect algebra and U be a nonempty set , $S_1, S_2, S_3, ... \subseteq L^U$, if $S_{1_+} = S_{2_+}$, $\widehat{S_2} = \widehat{S_3}$, $S_{3_+} = S_{4_+}$, ...

Then $\overline{S_1} = \overline{S_2} = \overline{S_3} = \overline{S_4} = \cdots$

Proof. By Lemma 3.13 and Corollary 3.14

$$\overline{S_1} = S_{1_\infty} = \widehat{\widehat{\widehat{S_{1_{+_{+_+}}}}}}_{\cdots}^{\cdots} = \widehat{\widehat{\widehat{S_{2_{+_{+_+}}}}}}_{\cdots}^{\cdots} = \overline{S_2} = \widehat{\widehat{\widehat{S_{2\,+\,+}}}}_{\cdots}^{\cdots} = \widehat{\widehat{\widehat{S_{3\,+\,+}}}}_{\cdots}^{\cdots} = \overline{S_3} = \cdots \tag{22}$$

Lemma Let L be an effect algebra and U be a nonempty set , $S_1, S_2, S_3, ... \subseteq L^U$, if $\widehat{S_1} = \widehat{S_2}$, $S_{2_+} = S_{3_+}$, $\widehat{S_3} = \widehat{S_4}$, ...

Then, $\overline{S_1} = \overline{S_2} = \overline{S_3} = \overline{S_4} = \cdots$

Proof. Similar to the previous Lemma.

The following result is obtained from Lemma 4.1

If S_1 is the input system of L-sets, we obtain (a smaller) S_2 with $S_{1_+} = S_{2_+}$ and then obtain from S_2 some smaller S_3 with $\widehat{S_2} = \widehat{S_3}$ and we obtain from S_3 some smaller S_4 with $S_{3_+} = S_{4_+}$,...

then $S_1, S_2, S_3, S_4, \ldots$ generate the same L -closure systems. In particular, in view of the preceding results, a good choice is to take $Rdu_+(S)$, $Rdu_\wedge(S)$, In this procedure, These considerations bring us a way to find the base of $\bar{S}$.

Corollary Let L be an effect algebra and U be a nonempty set. If $S \subseteq L^U$, then $Rdu_+(\ldots Rdu_+(Rdu_\wedge(Rdu_+(S))))$ or $Rdu_\wedge(\ldots Rdu_\wedge(Rdu_+(Rdu_\wedge(S))))$ is a base of $\bar{S}$.

future work In future work, we are looking for conditions in which $\bar{S}$ and bass for $\bar{S}$ can be operated with finite operations.

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